Star-based a posteriori error estimates for elliptic problems.

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Abstract. We give an a posteriori error estimator for nonconforming finite element approximations of diffusion-reaction and Stokes problems, which relies on the solution of local problems on stars. It is proved to be equivalent to the energy error up to a data oscillation, without requiring Helmholtz decomposition of the error nor saturation assumption. Numerical experiments illustrate the good behavior and efficiency of this estimator for generic elliptic problems.

Key words: A posteriori error estimator, nonconforming finite element method, diffusion reaction equations, Stokes equations.
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1. Introduction

During the last two decades a large amount of work has been devoted to a posteriori error estimation for solution approximated either by conforming [1, 17] or nonconforming [2, 8, 9] finite element methods. In the nonconforming context, two main approaches have been considered for constructing an a posteriori error estimator. In residual estimators some extra terms have to be added to well-known a posteriori error estimator used in conforming framework. In [3, 4, 10], these extra terms are the jumps across the element edges of the tangential derivatives of the finite element approximation with respect to element edges.

One of the most successful estimators proposed by Bank and Weiser and extended by many authors ([1, 6, 7, 12]), is based on the solution of local Neumann problems on elements, which seems to allow for cancellation and yields better effectivity indices than residual estimators in numerical tests performed in [11]. The classical proof of equivalence with the energy error require the *saturation assumption* which states that this solution can be approximated asymptotically better with quadratic than with linear finite elements. The saturation assumption is shown to be superfluous by Nochetto in [13]. However, removing this assumption requires comparison with residual estimators. More recently, an approach based on the solution of local problems on stars was proposed in [11, 15], and the proof of the equivalence with energy error applies directly without reference to residual estimators. This approach is applied in

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[5, 14] to nonconforming approximations of two-dimensional second order elliptic problems, where the equivalence between the exact error and the estimator on star, is based there on Helmholtz decomposition of the error, which is no more valid in general three dimensional geometries due to convexity requirement.

In this paper, an alternative approach for constructing an a posteriori error estimator for nonconforming approximation of scalar second order elliptic problem, based on the solution of local problems on stars, is given. We prove in general dimensions the efficiency and the reliability of this estimator, without saturation assumption. Moreover, explicit constants for transfer operator ([4, 16]) are given, which proves that this estimator is robust in suitable norms. The outline of the paper is as follows. In section 2 we introduce the functional framework and introduce the diffusion-reaction problem with non nonconforming finite element approximation. In section three, we introduce the star-based a posteriori error estimator and perform the analysis for diffusion reaction problem. In section four, we extend the analysis to Stokes equation by adapting introduced arguments. Numerical results are given in section five to illustrate the good behavior and the efficiency of the given estimator on examples involving smooth and less-smooth solutions.

2. Setting of the problem

We consider the diffusion-reaction problem

$$(P)\ \begin{cases} -\Delta u + \sigma u = f \ \ in \ \ \Omega, \\ u = 0 \ \ on \ \ \Gamma := \partial\Omega, \end{cases}$$

where we assume that $\sigma \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$, and $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a simply connected polygonal domain. Let $\mathcal{T}_h$ be a family of conforming shape-regular triangulations of Ω by d-simplexes. We denote by E_I the set of interior edges (faces) and by E_f the set of all edges (faces) included in Γ. Let V_h be the lowest order nonconforming Crouzeix-Raviart finite element space defined by

$$V_h = \{v_h \in L^2(\Omega); \forall T \in \mathcal{T}_h,\ v_{h|T} \in P_1(T), \forall E \in E_I,\ \int_E [v_h]_E d\gamma = 0 \text{ and } \forall E \in E_f,\ \int_E v_h d\gamma = 0\},$$

where $[.]_E$ denotes the jump of the argument across E.

We denote by $\{x_i\}_{i \in \mathcal{N}}$ the set of all nodes of the triangulation $\mathcal{T}_h$. In the paper, by $i \in \mathcal{N}$ we will refer to the node x_i. For each $i \in \mathcal{N}$, ϕ_i denotes the canonical continuous piecewise linear basis function associated to x_i. The star ω_i is the interior relative to Ω of the support of ϕ_i, and h_i is the maximal size (diameters) of the elements constituting ω_i. Finally, Γ_i denotes the union of the edges (faces) touching x_i that are contained in Ω, and $\overline{\Gamma}_i$ the union of the edges (faces) touching x_i that are contained in $\overline{\Omega}$. h_E denotes the size (diameter) of an edge (face) E.
For each star ω_i, $i \in \mathcal{N}$, we introduce the space $V(\omega_i)$ defined by

$$V(\omega_i) = \{v \in H^1_{loc}(\omega_i) : \int_{\omega_i} v\phi_i dx = 0\}, \qquad \text{if } x_i \text{ is an interior node,}$$

and

$$V(\omega_i) = \{v \in H^1_{loc}(\omega_i) : v = 0 \ \text{ on } \partial\omega_i \cap \Gamma\}, \qquad \text{if } x_i \text{ is a boundary node.}$$

There exists a constant C, only depending on the minimum angle of the triangulation but independent of the star being considered, such that (see Prop. 2.4 of [11]) :

$$\forall v \in V(\omega_i), \quad \|v\|_{0,\omega_i} \leq C h_i \Big(\int_{\omega_i} |\nabla v|^2 \phi_i dx \Big)^{1/2}. \tag{2.1}$$

We define the finite dimensional local spaces $\mathcal{P}^2(\omega_i)$ and $\mathcal{P}^2_0(\omega_i)$ as follows,

Definition 1. *For $i \in \mathcal{N}$, let $\mathcal{P}^2(\omega_i)$ denote the space of continuous piecewise quadratic functions on the star ω_i that vanish on $\partial\omega_i$. The space $\mathcal{P}_0^2(\omega_i)$ is defined by $\mathcal{P}_0^2(\omega_i) = \mathcal{P}^2(\omega_i) \cap V(\omega_i)$.*

Let us introduce the usual H^1-norm on ω_i,

$$||u||_{1,\omega_i}^2 = \|\nabla u\|_{0,\omega_i}^2 + \|u\|_{0,\omega_i}^2 .$$

Let $v_h \in V_h$ be fixed. We denote by $\nabla_h v_h$ the vector field belonging to $(L^2(\Omega))^d$, defined by

$$\forall T \in \mathcal{T}_h, \quad \nabla_h v_h = \nabla v_h \quad \text{on } T.$$

Let $u_h^{NC} \in V_h$ be a solution of the nonconforming approximation problem:

$$(P_h)^{NC} \quad \forall v_h \in V_h \cap H_0^1(\Omega), \qquad a(u_h^{NC}, v_h) := \sum_{T \in \mathcal{T}_h} \int_T [\nabla u_h^{NC}.\nabla v_h + \sigma u_h^{NC} v_h] dx = \int_\Omega f v_h dx.$$

We will now turn to the construction of a u_h^C belonging to $V_h \cap H_0^1(\Omega)$ such that

$$||u - u_h^C||_{1,\Omega} \simeq \sum_{i \in \mathcal{N}} (||u - u_h^{NC}||_{1,\omega_i}^2)^{\frac{1}{2}}.$$

3. The star-based error estimate

For each $i \in \mathcal{N}$, we consider the local problems :

$$(P_i) \quad \left\{ \begin{array}{l} \text{Find } \eta_i \in \mathcal{P}_0^2(\omega_i) \text{ such that } \forall \mu_i \in \mathcal{P}_0^2(\omega_i), \\ \displaystyle\int_{\omega_i} (\nabla \eta_i . \nabla \mu_i) \phi_i dx = \int_{\omega_i} \nabla_h u_h^{NC} . \nabla(\mu_i \phi_i) dx + \int_{\omega_i} \sigma u_h^{NC} \mu_i \phi_i dx - \int_{\omega_i} f \mu_i \phi_i dx. \end{array} \right.$$

Using Lax-Milgram Theorem, we can prove that each discrete problem (P_i) admits a unique solution η_i.

Now we introduce the local error indicators,

$$\forall i \in \mathcal{N}, \forall u_h^{NC} \in V_h, \;\; E_{1,i}^2(u_h^{NC}) = \int_{\omega_i} |\nabla \eta_i|^2 \phi_i dx.$$

and

$$\forall i \in \mathcal{N}, \forall u_h^{NC} \in V_h, \;\; E_{2,i}^2(u_h^{NC}) = \sum_{E \in \omega_i} h_E^{-1} ||[u_h^{NC}]_E||_{0,E}^2.$$

3.1. Upper bound

We consider first the upper bound of the error without oscillation, and we step the process to the main theorem by the following intermediate lemmas.
The first lemma is an adaptation of arguments given in [11] and so the proof will be skipped.

Lemma 2. *For all $i \in \mathcal{N}$, there exists an operator $\Pi_i : V(\omega_i) \longrightarrow \mathcal{P}_0^2(\omega_i)$, such that for any $v \in V(\omega_i)$ the following conditions hold :*

1. *For all edge (face) $E \subset \Gamma_i$,* $\displaystyle\int_E (v - \Pi_i v)\phi_i \, d\gamma = 0.$
2. *Moreover,* $\displaystyle\int_{\omega_i} (v - \Pi_i v)\phi_i \, dx = 0,$ *if x_i is an interior node.*
3. $\displaystyle\Big(\int_{\omega_i} |\nabla \Pi_i v|^2 \phi_i dx \Big)^{\frac{1}{2}} \leq C \Big(\int_{\omega_i} |\nabla v|^2 \phi_i dx \Big)^{\frac{1}{2}}.$

where C is a positive constant only depending on the minimum angle of $\mathcal{T}_h$.

Lemma 3. *For each $i \in \mathcal{N}$, each $v \in V(\omega_i)$ and $u_h \in V_h$, we have*

$$\int_{\omega_i} \nabla_h u_h . \nabla((\Pi_i v)\phi_i) dx = \int_{\omega_i} \nabla_h u_h . \nabla(v\phi_i) dx.$$

Proof. If we denote by $[\dfrac{\partial u_h}{\partial n_E}] \in P_0(E)$ the jump of the normal derivative across E, we have by applying Green formula and subsequently using the property 1. of Lemma 2,

$$\int_{\omega_i} \nabla_h u_h . \nabla((\Pi_i v)\phi_i) dx = \sum_{E \subset \omega_i} \int_E [\frac{\partial u_h}{\partial n_E}](\Pi_i v)\phi_i d\gamma = \sum_{E \subset \omega_i} \int_E [\frac{\partial u_h}{\partial n_E}] v \phi_i d\gamma. \tag{3.1}$$

Applying again Green formula yields the result. □

Now we define the data oscillation by

$$osc(f) = \Big(\sum_{i \in \mathcal{N}} h_i^2 \|(f - f_i - \sigma u_h^{NC})\phi_i^{\frac{1}{2}}\|_{0,\omega_i}^2 \Big)^{\frac{1}{2}},$$

where $f_i = \dfrac{\int_{\omega_i} f\phi_i dx}{\int_{\omega_i} \phi_i dx}$ for i interior nodes and 0 otherwise.
We have the following result about the a posteriori error estimate for any conforming approximation;

Theorem 4. *Let $u_h^{NC} \in V_h$ be a solution of (P_h^{NC}) and $u_h^C \in V_h \cap H_0^1(\Omega)$. There exists a positive constant C only depending on the minimum angle of $\mathcal{T}_h$ such that*

$$\|u - u_h^C\|_{1,\Omega} \quad \leq \quad C \left[\left(\sum_{i \in \mathcal{N}} E_{1,i}^2(u_h^{NC}) \right)^{\frac{1}{2}} + \left(\sum_{i \in \mathcal{N}} \|u_h^{NC} - u_h^C\|_{1,\omega_i}^2 \right)^{\frac{1}{2}} + osc(f) \right]. \tag{3.2}$$

Proof. Let v be an element of $H_0^1(\Omega)$ and set $\tilde{v} := \sum_{i \in \mathcal{N}} v_i \phi_i$, where $v_i = \dfrac{\int_{\omega_i} v\phi_i dx}{\int_{\omega_i} \phi_i dx}$ for interior nodes and 0 otherwise.

We have by adapting standard arguments used in the analysis of finite element approximation of finite approximation of elliptic problems and introducing u_h^{NC},

$$\|u - u_h^C\|_{1,\Omega} \leq C \sup_{v \in H_0^1(\Omega)} \frac{|a(u - u_h^{NC}, v) + a(u_h^{NC} - u_h^C, v)|}{\|v\|_{1,\Omega}}.$$

Since $\tilde{v} \in V_h \cap H_0^1(\Omega)$, $\quad a(u_h^{NC} - u, \tilde{v}) = 0$. This gives,

$$\begin{aligned} a(u_h^{NC} - u, v) &= a(u_h^{NC} - u, v - \tilde{v}), \\ &= \sum_{i \in \mathcal{N}} \left[\int_{\omega_i} \nabla_h u_h^{NC} . \nabla(v - \tilde{v}) dx + \int_{\omega_i} \sigma u_h^{NC}(v - \tilde{v}) dx - \int_{\omega_i} f(v - \tilde{v}) dx \right], \end{aligned}$$

Stating that $v - \tilde{v} = \sum_{i \in \mathcal{N}} (v - v_i)\phi_i$, and using $\sum_{i\in\mathcal{N}} \phi_i(x) = 1$ gives

$$a(u_h^{NC} - u, v) = \sum_{i\in\mathcal{N}} \left[\int_{\omega_i} \nabla_h u_h^{NC} . \nabla[(v - v_i)\phi_i] dx + \int_{\omega_i} \sigma u_h^{NC} (v - v_i)\phi_i dx - \int_{\omega_i} f(v - v_i)\phi_i dx \right].$$

Since $(v - v_i) \in V(\omega_i)$, adding and removing same quantities in the two last terms give

$$\begin{aligned} a(u_h^{NC} - u, v) &= \sum_{i\in\mathcal{N}} \left[\int_{\omega_i} \nabla_h u_h^{NC} . \nabla[\Pi_i(v - v_i)\phi_i] dx + \int_{\omega_i} \sigma u_h^{NC} \Pi_i(v - v_i)\phi_i dx - \int_{\omega_i} f \Pi_i(v - v_i)\phi_i dx \right] \\ &- \sum_{i\in\mathcal{N}} \int_{\omega_i} (f - \sigma u_h^{NC})(v - v_i - \Pi_i(v - v_i))\phi_i dx. \end{aligned}$$

Using the definition of local problems (P_i),

$$a(u_h^{NC} - u, v) = \sum_{i\in\mathcal{N}} \left[\int_{\omega_i} \nabla\eta_i . \nabla[\Pi_i(v - v_i)]\phi_i dx \right] - \sum_{i\in\mathcal{N}} \left[\int_{\omega_i} (f - \sigma u_h^{NC})(v - v_i - \Pi_i(v - v_i))\phi_i dx \right].$$

We now process successively with each term of the right-hand side. On one hand, using Cauchy-Schwarz and item 2. of Lemma 2 we have

$$\begin{aligned} \sum_{i\in\mathcal{N}} \left[\int_{\omega_i} \nabla\eta_i . \nabla\Pi_i(v - v_i)\phi_i dx \right] &\leq \Big(\sum_{i\in\mathcal{N}} \int_{\omega_i} |\nabla\eta_i|^2 \phi_i dx \Big)^{\frac{1}{2}} \Big(\sum_{i\in\mathcal{N}} \int_{\omega_i} |\nabla\Pi_i(v - v_i)|^2 \phi_i dx \Big)^{\frac{1}{2}}, \\ &\leq C\Big(\sum_{i\in\mathcal{N}} E_{1,i}^2(u_h^{NC}) \Big)^{\frac{1}{2}} \Big(\sum_{i\in\mathcal{N}} \int_{\omega_i} |\nabla(v - v_i)|^2 \phi_i dx \Big)^{\frac{1}{2}}, \\ &\leq C\Big(\sum_{i\in\mathcal{N}} E_{1,i}^2(u_h^{NC}) \Big)^{\frac{1}{2}} \|v\|_{1,\Omega}. \end{aligned}$$

On the other hand, since both of $(v - v_i)$ and $\Pi_i(v - v_i)$ belong to $V(w_i)$, using definition of $V(\omega_i)$ and coefficients f_i give

$$\sum_{i\in\mathcal{N}} \left[\int_{\omega_i} (f - \sigma u_h^{NC})(v - v_i - \Pi_i(v - v_i))\phi_i dx \right] = \sum_{i\in\mathcal{N}} \left[\int_{\omega_i} (f - f_i - \sigma u_h^{NC})(v - v_i - \Pi_i(v - v_i))\phi_i dx \right],$$

Using Cauchy-Schwarz then inequality (2.1) and once more $\sum_{i\in\mathcal{N}} \phi_i(x) = 1$, we get

$$\begin{aligned} \sum_{i\in\mathcal{N}} \left[\int_{\omega_i} (f - \sigma u_h^{NC})(v - v_i - \Pi_i(v - v_i))\phi_i dx \right] &\leq osc(f) (\sum_{i\in\mathcal{N}} h_i^{-2} \|(v - v_i - \Pi_i(v - v_i))(\phi_i)^{\frac{1}{2}}\|_{0,\omega_i}^2)^{\frac{1}{2}}, \\ &\leq C \, osc(f) \, \|v\|_{1,\Omega}. \end{aligned}$$

C is a generic constant only depending on the minimum angle of triangulation.
Finally, summing up the different contributions in the estimate of $\|u - u_h^C\|_{1,\Omega}$ and using the continuity of $a(.,.)$ yield the result. □

Summarizing the previous results gives the following result about the a posteriori error estimate for the nonconforming approximation:

Theorem 5. *Let $u_h^{NC} \in V_h$ be a solution of (P_h^{NC}) and u_h^C be an arbitrary function of $\in V_h \cap H_0^1(\Omega)$. We have*

$$\Big(\sum_{i\in\mathcal{N}} \|u - u_h^{NC}\|_{1,\omega_i}^2 \Big)^{\frac{1}{2}} \leq C \left[\Big(\sum_{i\in\mathcal{N}} \|u_h^{NC} - u_h^C\|_{1,\omega_i}^2 \Big)^{\frac{1}{2}} + \Big(\sum_{i\in\mathcal{N}} E_{1,i}^2(u_h^{NC}) \Big)^{\frac{1}{2}} + \ osc(f) \right], \tag{3.3}$$

where C only depends on the minimum angle of $\mathcal{T}_h$.

In order to prove now the reliability of the estimator, we need the following lemma [10],

Lemma 6. *There exists a linear operator I: $V_h \longrightarrow V_h \cap H_0^1(\Omega)$, satisfying the following estimate*

$$\forall u_h^{NC} \in V_h, \forall \omega_i \in \mathcal{T}_h, k=0,1, \;\; \|u_h^{NC} - Iu_h^{NC}\|_{k,\omega_i} \leq C \sum_{E\in E_I, \overline{E}\cap\omega_i \neq \emptyset} h_E^{\frac{1}{2}-k} \|[u_h^{NC}]_E\|_{0,E}, \tag{3.4}$$

where h_E is the diameter of face (edge) E.

3.2. Lower bound

In this section we prove a lower bound of the error without oscillation.

Theorem 7. *Let $u_h^{NC} \in V_h$, there exist generic positive constant C depending on the minimum angle of the triangulation such that, for any $i \in \mathcal{N}$,*

$$E_{1,i}(u_h^{NC}) \leq C||u - u_h^{NC}||_{1,\omega_i},$$

and

$$E_{2,i}(u_h^{NC}) \leq C(\sum_{i\in\mathcal{N}} \|u - u_h^{NC}\|^2_{1,\omega_i})^{\frac{1}{2}}.$$

Proof. We refer for a proof of second estimate to [10], and proceed with the first one. For each $i \in \mathcal{N}$, by definition of $E_{1,i}(u_h^{NC})$ and taking test function $\mu_i = \eta_i$ in local problem (P_i) give

$$\begin{aligned} E_{1,i}^2(u_h^{NC}) &= \int_{\omega_i} (|\nabla\eta_i|^2 \phi_i dx), \\ &= \int_{\omega_i} \nabla u_h^{NC}.\nabla(\eta_i\phi_i)dx + \int_{\omega_i} (\sigma u_h^{NC}\eta_i)\phi_i dx - \int_{\omega_i} f\eta_i\phi_i dx, \end{aligned} \tag{3.5}$$

Since $(\eta_i\phi_i) \in H_0^1(\omega_i)$, we have $\int_{\omega_i} \nabla u.\nabla(\eta_i\phi_i)dx + \int_{\omega_i} \sigma u\eta_i\phi_i dx = \int_{\omega_i} f\eta_i\phi_i dx$. This gives

$$\begin{aligned} E_{1,i}^2(u_h^{NC}) &= \int_{\omega_i} (\nabla_h u_h^{NC} - \nabla u)\nabla(\eta_i\phi_i)dx + \int_{\omega_i} (\sigma(u_h^{NC} - u)\eta_i\phi_i dx, \\ &= \int_{\omega_i} (\nabla_h u_h^{NC} - \nabla u)\nabla(\eta_i)\phi_i dx + \int_{\omega_i} (\nabla_h u_h^{NC} - \nabla u)(\eta_i)\nabla(\phi_i)dx + \int_{\omega_i} (\sigma(u_h^{NC} - u)\eta_i\phi_i dx. \end{aligned}$$

Applying Cauchy-Schwarz inequality gives

$$\begin{aligned} E_{1,i}^2(u_h^{NC}) \;\; \leq \;\; & \|u - u_h^{NC}\|_{1,\omega_i} E_{1,i}(u_h) + \|u - u_h^{NC}\|_{1,\omega_i} \|\eta_i\|_{0,\omega_i} \|\phi_i\|_{W^{1,\infty}(w_i)} \\ & + \|\sigma\|_{L^\infty(\omega_i)} \|u - u_h^{NC}\|_{0,\omega_i} \|\eta_i\|_{0,\omega_i}, \end{aligned}$$

ϕ_i being bounded in ω_i. Now since $\eta_i \in V(\omega_i)$, using (2.1), we have

$$\|\eta_i\|_{0,\omega_i} \leq C\; h_i E_{1,i}(u_h^{NC}).$$

Finally using the property $|\phi_i|_{W^{1,\infty}(\omega_i)} \leq \dfrac{C}{h_i}$, we get

$$E_{1,i}(u_h^{NC}) \leq C\Big(1 + \|\sigma\|_{L^\infty(\Omega)} h_i\Big) \|u - u_h^{NC}\|_{1,\omega_i}.$$

which concludes the proof. $\square$

4. Extension to Stokes problem

Let us now extend the ideas given above to the Stokes equations. We will define an error estimator for this problem and prove that it is equivalent with the energy error. Given a simply connected domain $\Omega \subset \mathbb{R}^d$ $d = 2, 3$, we consider then the Stokes problem,

$$(SP)\begin{cases} -\Delta u + \nabla p = f \quad \text{in } \Omega, \\ \\ \quad \text{div } u = 0 \quad \text{in } \Omega, \\ \\ \quad u = 0 \quad \text{on } \Gamma = \partial\Omega, \end{cases}$$

where $f \in (L^2(\Omega))^d$.

V_h being defined in section 2, we set

$$Q_h = \{q_h \in L_0^2(\Omega), q_{h|T} \in P_0(T), \forall T \in \omega_i \text{ and } \omega_i \in \mathcal{T}_h\}.$$

and consider the approximate solution $(u_h^{NC}, p_h) \in (V_h)^d \times Q_h$ defined by

$$\begin{cases} \forall v_h \in (V_h)^d, \quad \sum_{T\in\mathcal{T}_h} \Big\{ \int_T \nabla u_h^{NC} : \nabla v_h dx - \int_T p_h \text{div}\, v_h dx \Big\} = \int_\Omega f.v_h dx, \\ \\ \forall q_h \in Q_h, \quad \sum_{T\in\mathcal{T}_h} \int_T q \,\text{div}\, u_h^{NC} \, dx = 0 \end{cases}$$

Note that the second equation means that for every $T \in \mathcal{T}_h$, div $(u_h^{NC}{}_{|T}) = 0$.

Let $v_h \in (V_h)^d$ be fixed. We define $\nabla_h v_h$ and $\text{div}\,_h v_h$ by :

$$\forall T \in \mathcal{T}_h, \quad \nabla_h v_h = \nabla v_h \quad \text{on } T,$$

and

$$\forall T \in \mathcal{T}_h, \quad \text{div}\,_h v_h = \text{div}\, v_h \quad \text{on } T.$$

We introduce the following local problems :

$$(SP_i)\begin{cases} \text{Find } \epsilon_i \in (\mathcal{P}_0^2(\omega_i))^d \text{ such that} \\ \\ \forall \mu_i \in (\mathcal{P}_0^2(\omega_i))^d, \ \int_{\omega_i} (\nabla\epsilon_i : \nabla\mu_i)\phi_i dx = \int_{\omega_i} \nabla_h u_h^{NC} : \nabla(\mu_i\phi_i) dx \\ \\ \qquad - \int_{\omega_i} p_h \text{div}\,(\mu_i\phi_i) dx - \int_{\omega_i} (f.\mu_i)\phi_i dx. \end{cases}$$

It is obvious that these local problems admit unique solutions.

We introduce for all $i \in \mathcal{N}$ the three indicators,

$$\eta_{1,i}(u_h^{NC}, p_h) = \Big(\sum_{T\in\omega_i} \|\text{div}\,_h u_h^{NC} \phi_i^{\frac{1}{2}} \|_{0,T}^2 \Big)^{\frac{1}{2}},$$

$$\eta_{2,i}(u_h^{NC},p_h) = (\int_{\omega_i} |\nabla \epsilon_i|^2 \phi_i dx)^{\frac{1}{2}},$$

$$\eta_3^2(u_h^{NC},p_h) = \sum_{E\in E_I} h_E^{-1} ||[u_h^{NC}]_E||_{0,E}^2,$$

and set the problem data oscillation,

$$osc(f) = (\sum_{i\in\mathcal{N}} h_i^2 \|(f-f_i)\phi_i^{\frac{1}{2}}\|_{0,\omega_i}^2)^{\frac{1}{2}},$$

where $f_i = \dfrac{\int_{\omega_i} f\phi_i dx}{\int_{\omega_i}\phi_i dx}$ for interior nodes, and $f_i = 0$ otherwise.

As previously, we give the first lemma which proof is similar to Lemma 2 one.

Lemma 8. *For each $i \in \mathcal{N}$, there exists an operator $\Pi_i : (V(\omega_i))^d \longrightarrow (\mathcal{P}_0^2(\omega_i))^d$, such that for any $v \in (V(\omega_i))^d$ the following assumptions hold :*

1. *For all edge $E \subset \Gamma_i$, $\displaystyle\int_E (v-\Pi_i v)\phi_i d\gamma = 0$,*
2. *For all $v \in (V(\omega_i))^d$ and $v_h \in (V_h)^d$, $\displaystyle\int_{\omega_i} \nabla_h v_h : \nabla((\Pi_i v - v)\phi_i)dx = 0$,*
3. *For all $q_h \in Q_h$, $\displaystyle\int_{\omega_i} q_h \mathrm{div}\,(v-\Pi_i v)\phi_i dx = 0$.*

The following theorem gives the a posteriori error estimate for the nonconforming finite element approximation of Stokes problem solution.

Theorem 9. *There exists a positive constant C depending on the minimum angle of the triangulation such that :*

$$\Big(\sum_{i\in\mathcal{N}} \|u-u_h^{NC}\|_{1,\omega_i}^2\Big)^{\frac{1}{2}} + \|p-p_h\|_{0,\Omega} \quad\leq\quad C\left\{\Big[\Big(\sum_{i\in\mathcal{N}} \eta_{1,i}^2 + \eta_{2,i}^2\Big) + \eta_3^2\Big]^{\frac{1}{2}} + osc(f)\right\}, \tag{4.1}$$

where, for more readability, we have skipped the arguments of $\eta_{1,i}$, $\eta_{2,i}$ and η_3, and so will be done in the sequel.

Proof. Since $(Iu_h^{NC},p_h) \in (H_0^1(\Omega))^d \times L_0^2(\Omega)$, by standard finite element analysis arguments we state

$$\Big(\sum_{i\in\mathcal{N}} \|u-Iu_h^{NC}\|_{1,\omega_i}^2\Big)^{\frac{1}{2}} + \|p-p_h\|_{0,\Omega} \leq C \sup_{(v,q)\in(H_0^1(\Omega)))^d\times L_0^2(\Omega))} \frac{|a((u,p);(v,q)) - a((Iu_h^{NC},p_h);(v,q))|}{|v|_{1,\Omega}+\|q\|_{0,\Omega}},$$

where $a(.;.)$ is defined by

$$\forall (u,p),(v,q) \in (H_0^1(\Omega))^d \times L_0^2(\Omega),\ a((u,p);(v,q)) = \int_\Omega \nabla u : \nabla v\, dx - \int_\Omega p\,\mathrm{div}\, v\, dx + \int_\Omega q\,\mathrm{div}\, u\, dx.$$

then

$$a((u,p);(v,q)) - a((Iu_h^{NC},p_h);(v,q)) = \int_\Omega (\nabla u - \nabla I u_h^{NC}) : \nabla v\, dx - \int_\Omega (p-p_h)\,\mathrm{div}\, v dx + \int_\Omega q\,\mathrm{div}_h(u - Iu_h^{NC})\, dx.$$

On one hand, since $\operatorname{div} u = 0$ on Ω, $\sum_i \phi_i(x) = 1$ and ϕ_i being bounded, we have

$$| \int_\Omega q \operatorname{div}(u - Iu_h^{NC})\, dx| \le C \sum_{\omega_i \in \mathcal{T}_h} \|q\|_{0,\omega_i} \|\operatorname{div} Iu_h^{NC} \phi_i^{\frac{1}{2}}\|_{0,\omega_i}.$$

By virtue of Lemma 6 with $k = 1$, we have

$$\begin{aligned} \forall \omega_i \in \mathcal{T}_h,\ \|\operatorname{div} Iu_h^{NC} \phi_i^{\frac{1}{2}}\|_{0,\omega_i} &\le \|\operatorname{div}_h u_h^{NC} \phi_i^{\frac{1}{2}}\|_{0,\omega_i} + \|(\operatorname{div}_h u_h^{NC} - \operatorname{div} Iu_h^{NC})\phi_i^{\frac{1}{2}}\|_{0,\omega_i}, \\ &\le \|\operatorname{div}_h u_h^{NC}\|_{0,\omega_i} + C \sum_{E \in E_I} h^{-\frac{1}{2}} \|[u_h^{NC}]_E\|_{0,E}. \end{aligned}$$

Summing up the contributions, using given indicators definitions and the inequality $\sum_i \alpha_i \beta_i \le (\sum_i \alpha_i^2)^{1/2} (\sum_i \beta_i^2)^{1/2}$, we get

$$\Big| \int_\Omega q \operatorname{div}(u - Iu_h^{NC})\, dx \Big| \le C \Big(\sum_{\omega_i \in \mathcal{T}_h} \|q\|_{0,\omega_i}^2 \Big) \Big(\sum_{i \in \mathcal{N}} \eta_{1,i}^2 + \eta_3^2 \Big)^{\frac{1}{2}}.$$

On the other hand,

$$\begin{aligned} A : &= \int_\Omega (\nabla u - \nabla I u_h^{NC}) : \nabla v\, dx - \int_\Omega (p - p_h) \operatorname{div} v\, dx = \sum_{\omega_i \in \mathcal{T}_h} \Big[\int_{\omega_i} (\nabla u - \nabla_h u_h^{NC}) : \nabla v\, dx \\ &\quad + \int_{\omega_i} (\nabla_h u_h^{NC} - \nabla I u_h^{NC}) : \nabla v\, dx - \int_{\omega_i} (p - p_h) \operatorname{div} v\, dx \Big], \\ &= -\sum_{\omega_i \in \mathcal{T}_h} \int_{\omega_i} (\nabla_h u_h^{NC}) : \nabla v\, dx + \int_{\omega_i} p_h \operatorname{div} v\, dx + \int_{\omega_i} f.v\, dx + \int_{\omega_i} (\nabla_h u_h^{NC} - \nabla I u_h^{NC}) : \nabla v\, dx, \end{aligned}$$

Introducing the field $\tilde{v} \in (V_h)^d \cap (H_0^1(\Omega))^d$, in the same manner as in proof of theorem 4 in order to involve $(v - v_i) \in (V(\omega_i))^d$ and use item 2. of Lemma 8 we get

$$\begin{aligned} A &= -\sum_{\omega_i \in \mathcal{T}_h} \int_{\omega_i} (\nabla_h u_h^{NC}) : \nabla[\Pi_i(v - v_i)\phi_i]\, dx + \int_{\omega_i} p_h [\operatorname{div} \Pi_i(v - v_i)\phi_i]\, dx + \int_{\omega_i} f.\Pi_i(v - v_i)\phi_i\, dx \\ &+ \int_{\omega_i} p_h [\operatorname{div}_h (v - v_i - \Pi_i(v - v_i))\phi_i]\, dx + \int_{\omega_i} (\nabla_h u_h^{NC} - \nabla I u_h^{NC}) : \nabla v\, dx + \int_{\omega_i} f.(v - v_i - \Pi_i(v - v_i))\phi_i\, dx, \end{aligned}$$

Adapting arguments used in Theorem 4 and using successively item 3. of Lemma 8, definition of local problems (SP_i), Lemma 6 with $k = 1$, we get

$$A \le \Big\{ \Big(\sum_{i \in \mathcal{N}} \eta_{2,i}^2 + \eta_3^2 \Big)^{\frac{1}{2}} + osc(f) \Big\} |v|_{1,\Omega}.$$

Summing up the contributions gives,

$$|a((u,p);(v,q)) - a((Iu_h^{NC}, p_h);(v,q))| \le \Big(\sum_{i \in \mathcal{N}} \eta_{1,i}^2 + \eta_3^2 \Big)^{\frac{1}{2}} \|q\|_{0,\Omega} + \Big\{ \Big(\sum_{i \in \mathcal{N}} \eta_{2,i}^2 + \eta_3^2 \Big)^{\frac{1}{2}} + osc(f) \Big\} |v|_{1,\Omega}. \quad (4.2)$$

Finally stating

$$\sum_{\omega_i \in \mathcal{T}_h} \|u - u_h^{NC}\|_{1,\omega_i}^2 \le \sum_{\omega_i \in \mathcal{T}_h} \|u - Iu_h^{NC}\|_{1,\omega_i}^2 + \sum_{\omega_i \in \mathcal{T}_h} \|u_h^{NC} - Iu_h^{NC}\|_{1,\omega_i}^2 \le C\eta_3^2 + \sum_{\omega_i \in \mathcal{T}_h} \|u - Iu_h^{NC}\|_{1,\omega_i}^2,$$

yield the result.

Efficiency of the estimator:

Theorem 10. $\forall \omega_i \in \mathcal{T}_h$, *we have the following inequalities,*

$$\|\mathrm{div}\,_h u_h^{NC}\|_{0,\omega_i} \leq C\|u - u_h^{NC}\|_{1,\omega_i}, \tag{4.3}$$

$$\eta_{2,i} \leq C\Big(\|u - u_h^{NC}\|_{1,\omega_i} + \|p - p_h\|_{0,\omega_i}\Big) \tag{4.4}$$

and

$$\eta_3 \leq C\|u - u_h^{NC}\|_{1,\omega_i}. \tag{4.5}$$

Proof. The first inequality is obvious and the third one has already been proved in Theorem 7. So we proceed with the second estimation. Using the definition indicator and local problems (SP_i),

$$\begin{aligned}
\eta_{2,i}^2(u_h^{NC}, p_h) &= \int_{\omega_i} (|\nabla \epsilon_i|^2 \phi_i dx), \\
&= \int_{\omega_i} \nabla_h u_h^{NC} : \nabla(\epsilon_i \phi_i) dx - \int_{\omega_i} p_h \mathrm{div}\,(\epsilon_i \phi_i) dx - \int_{\omega_i} f.\epsilon_i \phi_i dx,
\end{aligned}$$

As in section 3, since $(\epsilon_i \phi_i) \in (H_0^1(\omega_i))^d$, we can state

$$\begin{aligned}
\eta_{2,i}^2(u_h^{NC}, p_h) &= \int_{\omega_i} (\nabla_h u_h^{NC} - \nabla u) : \nabla(\epsilon_i \phi_i) dx - \int_{\omega_i} (p_h - p) \mathrm{div}\,(\epsilon_i \phi_i) dx, \\
&= \int_{\omega_i} (\nabla_h u_h^{NC} - \nabla u) : \nabla(\epsilon_i) \phi_i dx + \int_{\omega_i} (\nabla_h u_h^{NC} - \nabla u)(\epsilon_i) : \nabla(\phi_i) dx - \int_{\omega_i} (p_h - p) \mathrm{div}\,(\epsilon_i \phi_i) dx,
\end{aligned}$$

and following same steps as in proof of Theorem 7, we retrieve the second estimation.

5. Numerical experiments

Diffusion reaction example:

For the numerical illustration of the efficiency of the error estimator and the based adaption process, we consider a model problem with homogeneous data on the computational domain $[0,1]^2$, with the source term f given by the exact solution,

$$u = xy(x-1)(y-1)e^{-100(x-0.5)^2 - 100(y-0.117)^2},$$

which presents sharp curvature in the vicinity of point $(0.5, 0.117)$, and we perform a nonconforming finite element discretization on it. Successive iterations of adaptive mesh are represented in Figure 1. Computed and Exact solution are given in Figure 2, where the scaling of the height is the same for both pictures.

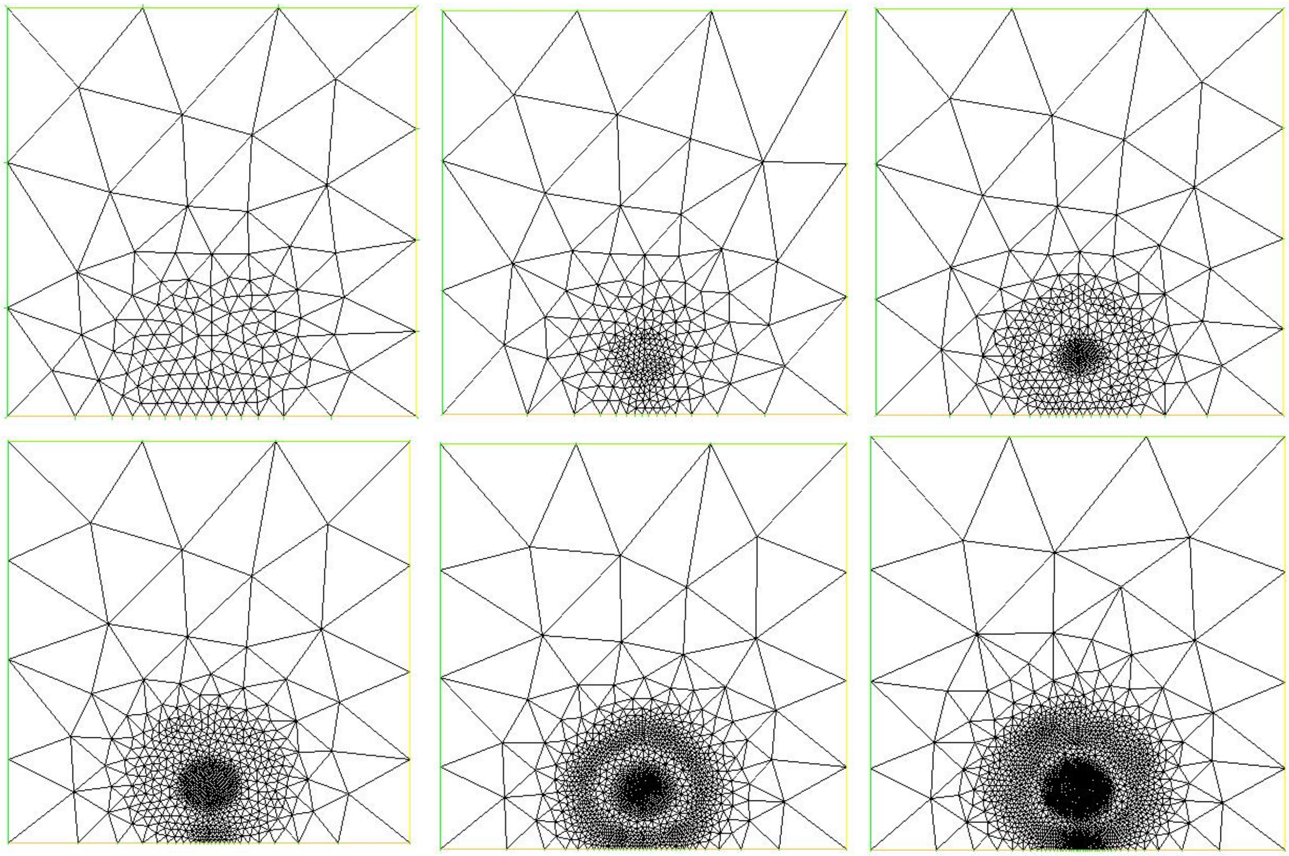

Figure 1: Adaptive mesh refinement using the error indicator.

Table 1 and Figure 3 give the evolution of the error indicator value and the error solution versus the number of degrees of freedom (ndof). We notice that the estimator and the error have analogous behavior, and the estimator under-estimates the energy norm error. Figure 3 illustrates quasi-optimality of the estimator, the dashed line of slope $(-1/2)$ showing a numerical (ndof)$^{(-1/2)}$ asymptotic decay of the error estimator.

Algorithm 1 Based adaption procedure

1: Generate an initial mesh and compute the solution.
2: **loop**
3: Calculate local error indicators and their sum.
4: Refine the mesh in the areas where the indicators are bigger than their mean value and compute solution.
5: If stopping criterium is satisfied, then exit the loop .
6: **end loop**

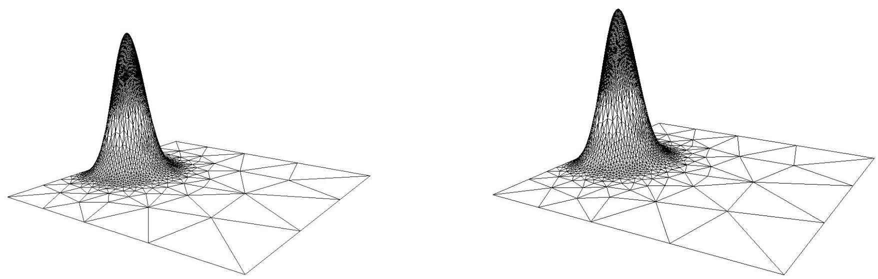

Figure 2: Computed solution (left) and exact solution (right) for diffusion reaction example (same scaling).

ndof	Error indicator	$\|u - u_h^{NC}\|$
449	3.4773e-001	1.6271e-001
762	2.5113e-001	1.1980e-001
1408	1.7055e-001	9.4704e-002
5065	8.8906e-002	5.3381e-002
9843	6.5920e-002	3.6450e-002
19576	4.7111e-002	2.8596e-002
38653	3.5282e-002	1.9497e-002
77469	2.5526e-002	1.5080e-002
153644	1.8019e-002	1.0170e-002

Table 1: Error and indicator values for diffusion reaction problem.

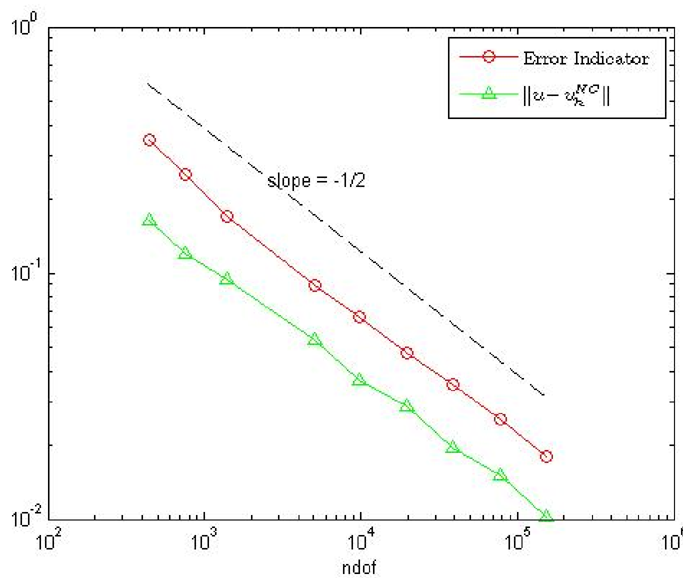

Figure 3: Decay of error indicator and energy error. The dashed line has slope of -1/2.

Stokes problem example with analytic smooth solution :

We consider the test case proposed par Bercovier and Engelman [18], defined on the unit square $[0, 1]^2$ as follows,

$$v(x, y) = -256x^2(x-1)^2y(y-1)(2y-1)$$

$$u(x,y) = \left[\begin{array}{c} v(x,y) \\ -v(x,y) \end{array} \right]$$

$$p(x,y) = (x-\frac{1}{2}).(y-\frac{1}{2})$$

$$f(x,y) = \left[\begin{array}{c} -\nu v(x,y) + (y-\frac{1}{2}) \\ \nu v(x,y) + (x-\frac{1}{2}) \end{array} \right]$$

We perform nonconforming finite element discretization on it, and we report on Figure 4, Figure 5 and Figure 6 a sequence of adapted meshes using the proposed refinement indicators and corresponding computed velocity and pressure respectively.

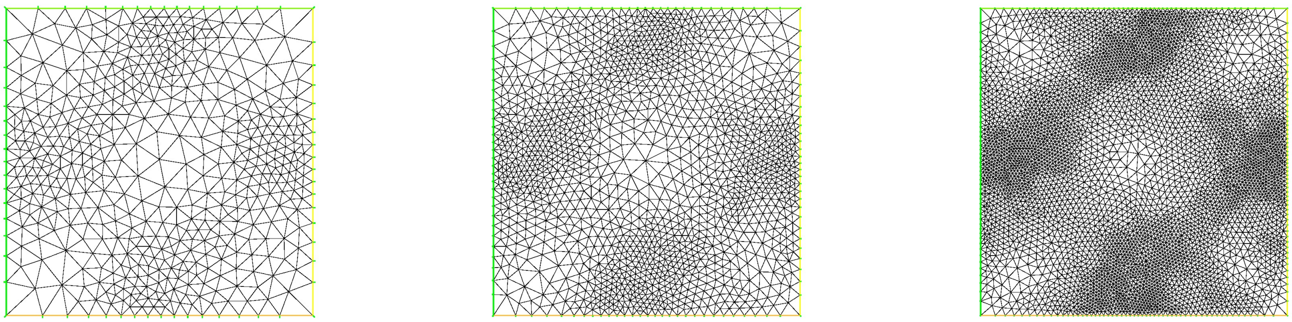

Figure 4: Adaptive mesh refinement using the error indicator.

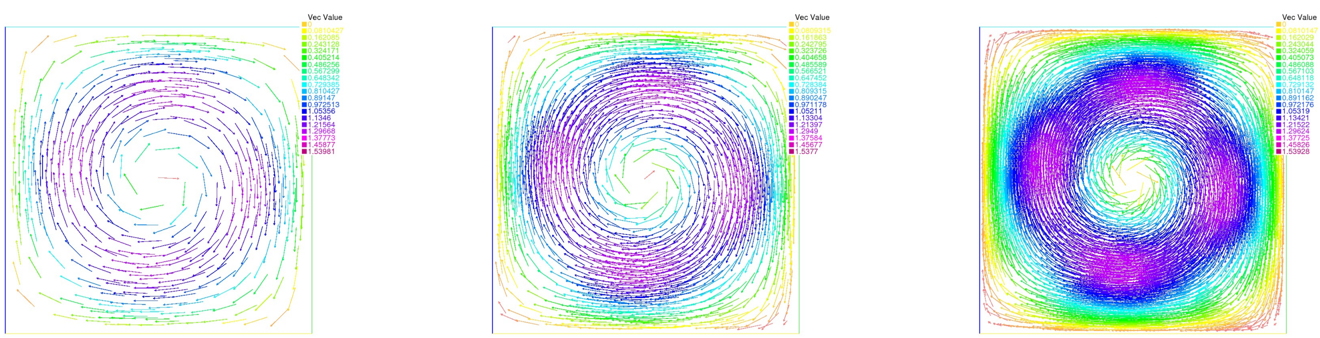

Figure 5: Adaptive computed velocity using the error indicator.

Lid-driven cavity problem example:

The two-dimensional Stokes driven cavity problem has been thoroughly studied in numerous references (eg. [19]). The main difficulty of this problem comes from the discontinuity of the velocity boundary data at corners. The problem configuration corresponds to a flow in a square cavity $[0,1]^2$.The top of the cavity moves from left to right, imparting motion to the fluid via the no-slip boundary condition, $\mathbf{u} = (1,0)$ on the top. The velocity on all other boundaries is zero, $\mathbf{u} = (0,0)$. We perform nonconforming finite element discretization on it, and we give below a sequence of adaptive meshes. Furthermore, we present the corresponding approximate velocity and pressure contour lines.

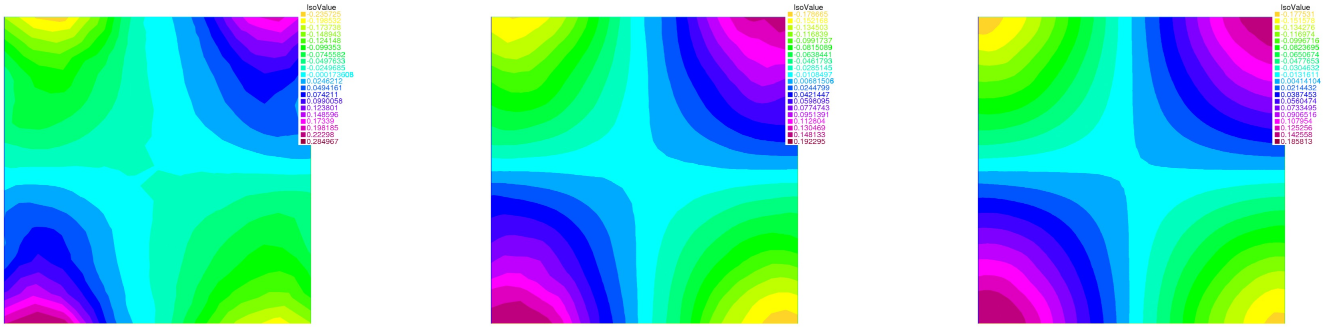

Figure 6: Adaptive computed pressure using the error indicator.

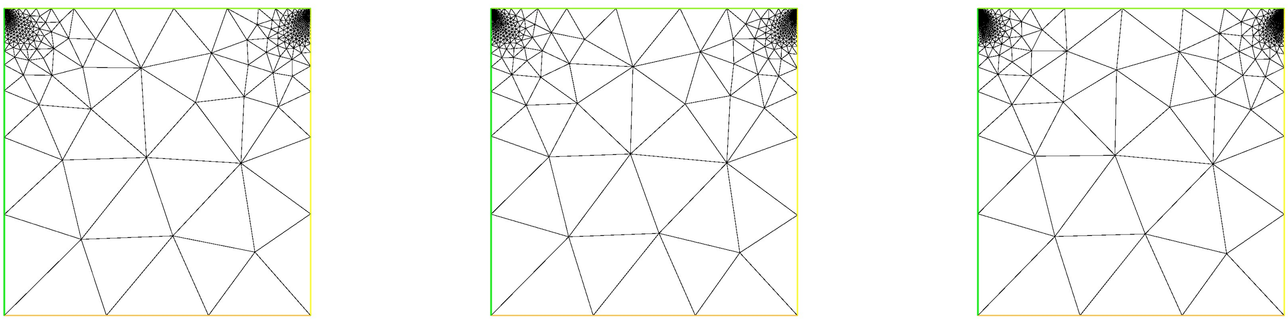

Figure 7: Error-indicator based refined meshes for Lid-driven Cavity problem.

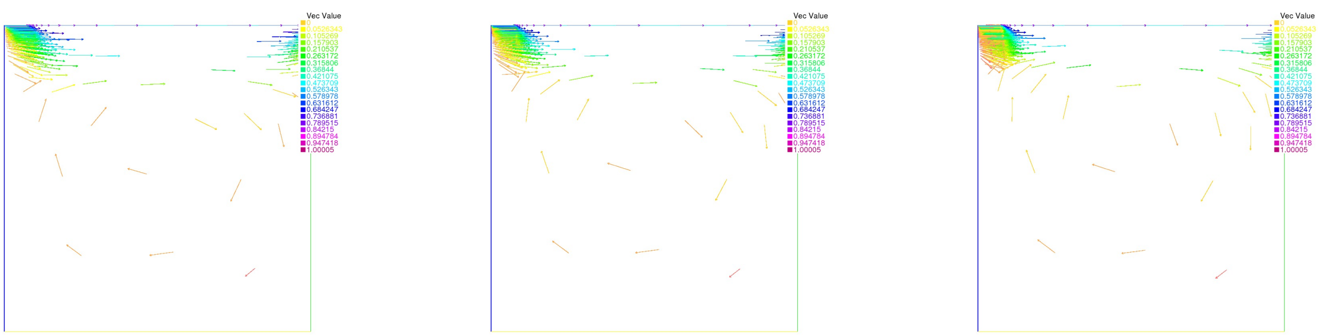

Figure 8: Adaptive computed velocity of Lid-driven Cavity problem.

6. Conclusion

We presented and analyzed an a posteriori error estimator for nonconforming approximations of reaction diffusion and Stokes equations. The construction of this so-called star-based error estimator is based on the solution of local sub-problems. We proved that it is equivalent to the energy error up to a data oscillation, without requiring Helmholtz decomposition of the error nor saturation assumption. The proof is valid in general space dimensions. Two-dimensional numerical experiments illustrated the good behavior and confirmed the quasi-optimal predicted asymptotic rate of decay of this error estimator.

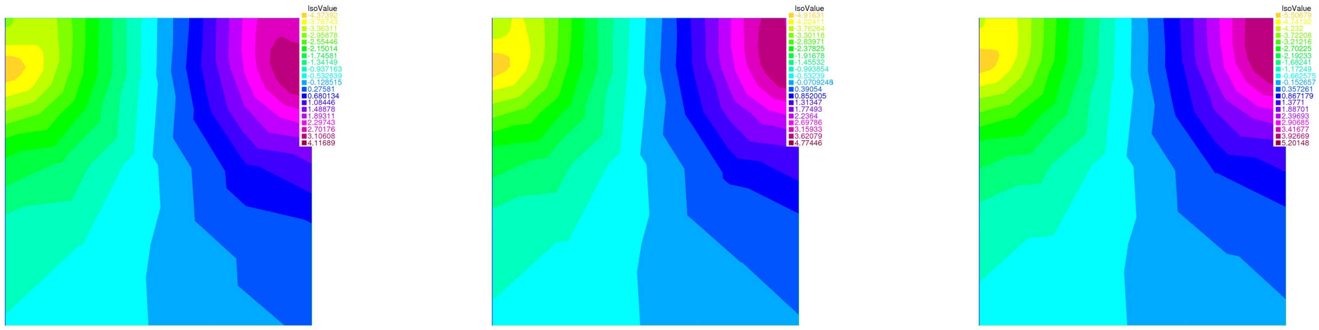

Figure 9: Adaptive computed pressure of Lid-driven Cavity problem.

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